Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel International Advanced Level

Friday 12 January 2024

Morning (Time: 1 hour 30 minutes) | **Paper reference** **WFM01/01**

Mathematics

International Advanced Subsidiary/ Advanced Level

Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over

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Pearson

1.

$$\mathbf{M} = \begin{pmatrix} 2k+1 & k \\ k+7 & k+4 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Show that **M** is non-singular for all real values of k.

(3)

(b) Determine $\mathbf{M}^{-1}$ in terms of k.

(2)

a. Matrix M is non-singular $\Rightarrow$ $\det(M) \neq 0$

☆ we are trying to show that $\det(M) \neq 0$ for all $k \in \mathbb{R}$ ☆

$\therefore$ start by finding $\det(M)$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(X) = (a)(d) - (b)(c)$

$\det(M) = (2k+1)(k+4) - (k)(k+7)$

$= (2k^2 + 9k + 4) - (k^2 + 7k)$

$= 2k^2 + 9k + 4 - k^2 - 7k$

$= k^2 + 2k + 4$

$\det(M) = k^2 + 2k + 4$

$= (k+1)^2 - 1 + 4$

$= (k+1)^2 + 3$

complete the square

Notice that all for all values of k, positive or negative, $(k+1)^2 \geq 0$. $\therefore (k+1)^2 + 3 \geq 3$

The smallest value of $\det(M) = 3$ when $k = -1$

$\therefore \det(M)$ is non-singular for all real values of k.

b. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X^{-1} = \dfrac{1}{\det(X)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$-b = -(k) = -k$

$-c = -(k+7) = -k-7$

$$M^{-1} = \frac{1}{k^2+2k+4} \begin{bmatrix} k+4 & -k \\ -k-7 & 2k+1 \end{bmatrix}$$

2.

$$f(z) = 2z^3 + pz^2 + qz - 41$$

where p and q are integers.

The complex number $5 - 4i$ is a root of the equation $f(z) = 0$

(a) Write down another complex root of this equation. **(1)**

(b) Solve the equation $f(z) = 0$ completely. **(4)**

(c) Determine the value of p and the value of q. **(2)**

When plotted on an Argand diagram, the points representing the roots of the equation $f(z) = 0$ form the vertices of a triangle.

(d) Determine the area of this triangle. **(2)**

a. For a cubic eq^n with form: $ax^3 + bx^2 + cx + d$, there are 3 roots.
There are 2 possibilities: 3 real roots
1 real root and 2 complex roots (conjugate pair)

The other complex root must be the complex conjugate of $(5-4i)$
flip sign of imaginary component

$\boxed{5+4i}$

b. $(z-(5-4i))(z-(5+4i))(z-\gamma)$, where γ is a root

$(z-5+4i)(z-5-4i)(z-\gamma)$

$(z^2 - 5z - \cancel{4iz} - 5z + 25 + \cancel{20i} + \cancel{4iz} - \cancel{20i} - 16i^2)(z-\gamma)$

$(z^2 - 10z + 25 - 16(-1))(z-\gamma)$

$(z^2 - 10z + 41)(z-\gamma)$

$z^3 - 10z^2 + 41z - \gamma z^2 + 10\gamma z - 41\gamma$

$(1) z^3 + (-10-\gamma)z^2 + (41+10\gamma)z + (-41\gamma)$ $\times 2$

$(2) z + (-20-2\gamma)z^2 + (82+20\gamma)z + (-82\gamma)$

P 7 4 3 0 9 A 0 4 3 2

Question 2 continued

equate coefficients to $f(z)$: (1) $2 = 2$

(2) $-20 - 2\gamma = p$

(3) $82 + 20\gamma = q$

(4) $-82\gamma = -41$

use (4) to solve γ: $-82\gamma = -41$

$\gamma = \frac{-41}{-82}$

$\gamma = \frac{1}{2}$

$\boxed{z = 5 \pm 4i, \frac{1}{2}}$

c. sub $\gamma = \frac{1}{2}$ back into (2) and (3) to find p and q

(3) $-20 - 2\gamma = p$

$-20 - 2\left(\frac{1}{2}\right) = p$

$p = -21$

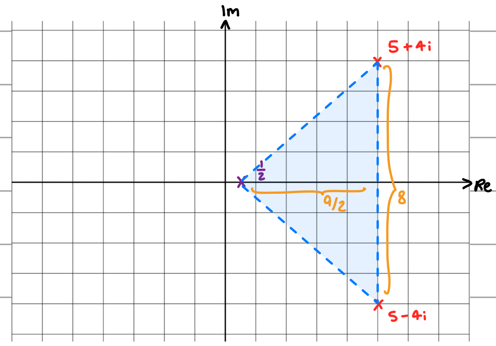
(4) $82 + 20\gamma = q$

$82 + 20\left(\frac{1}{2}\right) = q$

$q = 92$

$\boxed{p = -21, q = 92}$

d.



(NOT DRAWN TO SCALE)

$\Delta AOT = \frac{1}{2} \times \frac{9}{2} \times 8 = 18$

$\boxed{\Delta AOT = 18 \text{ units}^2}$

3. The hyperbola H has equation $xy = c^2$ where c is a positive constant.

The point $P\left(ct, \frac{c}{t}\right)$, where $t > 0$, lies on H.

The tangent to H at P meets the x-axis at the point A and meets the y-axis at the point B.

(a) Determine, in terms of c and t,

(i) the coordinates of A,

(ii) the coordinates of B.

(4)

Given that the area of triangle AOB, where O is the origin, is 90 square units,

(b) determine the value of c, giving your answer as a simplified surd.

(2)

a. Firstly, find tangent to H @ point P:

Differentiate rectangular hyperbola eqn w.r.t. x:

$xy = c^2$

$y = \frac{c^2}{x} = c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2} = \frac{-c^2}{x^2}$

Sub in point P $(ct, \frac{c}{t})$:

$\left.\frac{dy}{dx}\right|_{(ct, \frac{c}{t})} \frac{-c^2}{(ct)^2} = \frac{-c^2}{c^2t^2} = \frac{-1}{t^2}$

$M_{tangent} = -\frac{1}{t^2}$

set up line eqn $y - y_1 = m(x - x_1)$ with $m = \frac{-1}{t^2}$

$(x_1, y_1) = (ct, \frac{c}{t})$

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2(y - \frac{c}{t}) = -(x - ct)$

$t^2 y - ct = -x + ct$

$t^2 y + x = 2ct$ ☆

i. @ x-axis, $y = 0$, so to find when tangent crosses x-axis, set $y = 0$

$t^2(0) + x = 2ct$

$x = 2ct$

$\boxed{A\ (2ct, 0)}$

Question 3 continued

ii. @ y-axis, $x = 0$, so to find when tangent crosses y-axis, set $x = 0$

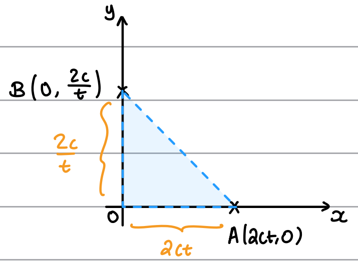
$t^2y + (0) = 2ct$

$t^2y = 2ct$

$y = \frac{2ct}{t^2} = \frac{2c}{t}$

$\boxed{\therefore B\left(0, \frac{2c}{t}\right)}$

b. do a sketch:



Area of Triangle AOB: $\cancel{\frac{1}{2}} \times (\cancel{2}c\cancel{t}) \times \left(\frac{2c}{\cancel{t}}\right) = 90$

$\therefore 2c^2 = 90$

$\therefore c^2 = 45$

$\therefore c = \pm\sqrt{45} = \pm 3\sqrt{5}$

However Q states c is a positive constant $\therefore c = 3\sqrt{5}$ only

$\boxed{c = 3\sqrt{5}}$

(Total for Question 3 is 6 marks)

4.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

(a) Describe the single geometrical transformation represented by the matrix **A**. **(2)**

The matrix **B** represents a rotation of 210° anticlockwise about centre (0, 0).

(b) Write down the matrix **B**, giving each element in exact form. **(1)**

The transformation represented by matrix **A** followed by the transformation represented by matrix **B** is represented by the matrix **C**.

(c) Find **C**. **(2)**

The hexagon H is transformed onto the hexagon H' by the matrix **C**.

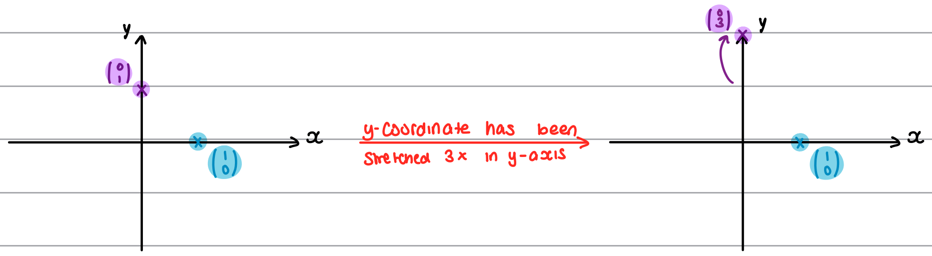
(d) Given that the area of hexagon H is 5 square units, determine the area of hexagon H' **(2)**

a. Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

then mark the coordinates of new matrix

identity matrix I　　　　　　　　　　　　matrix A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{transformation}} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$



$\boxed{\text{Stretch scale factor 3 parallel to the y-axis}}$

b.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ ← Anticlockwise rotation through θ about O (Given in Formulae booklet)

sub in $\theta = 210°$ → $\begin{bmatrix} \cos(210°) & -\sin(210°) \\ \sin(210°) & \cos(210°) \end{bmatrix} \rightarrow \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

Question 4 continued

$$\boxed{\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}}$$

c. order : $C = BA$

$$C = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} (-\frac{\sqrt{3}}{2})(1) + (\frac{1}{2})(0) & (-\frac{\sqrt{3}}{2})(0) + (\frac{1}{2})(3) \\ (-\frac{1}{2})(1) + (-\frac{\sqrt{3}}{2})(0) & (-\frac{1}{2})(0) + (-\frac{\sqrt{3}}{2})(3) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{3\sqrt{3}}{2} \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}}$$

d. $|\det(M)|$ = area scale factor

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det(X) = (a)(d) - (b)(c)$

$$\det(C) = \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)$$

$= 3$

$\det(C) = 3$

$|\det(C)| = |3| = 3$

Area of H $\times$ 3 = Area of H'

5 $\times$ 3 = 15

$\boxed{\text{Area of } H' = 15 \text{ units}^2}$

(Total for Question 4 is 7 marks)

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5. The quadratic equation

$$2x^2 - 3x + 7 = 0$$

has roots α and β

Without solving the equation,

(a) write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$

(1)

(b) determine the value of $\alpha^2 + \beta^2$

(2)

(c) find a quadratic equation which has roots

$$\left(\alpha - \frac{1}{\beta^2}\right) \text{ and } \left(\beta - \frac{1}{\alpha^2}\right)$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

(6)

a. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

let α, β be roots of quadratic

$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$

sum of roots : $\alpha + \beta$

product of roots: $\alpha\beta$

$2x^2 - 3x + 7 = 0$ ÷2

$x^2 - \frac{3}{2}x + \frac{7}{2} = 0$

$\alpha + \beta = \frac{3}{2}$

$\alpha\beta = \frac{7}{2}$

$\boxed{\alpha + \beta = \frac{3}{2} \quad \alpha\beta = \frac{7}{2}}$

b. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(\frac{3}{2}\right)^2 - 2\left(\frac{7}{2}\right)$

$= -\frac{19}{4}$

$\boxed{\alpha^2 + \beta^2 = -\frac{19}{4}}$

Question 5 continued

c. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$$x^2 - \left(\alpha - \frac{1}{\beta^2} + \beta - \frac{1}{\alpha^2}\right)x + \left(\left(\alpha - \frac{1}{\beta^2}\right) \times \left(\beta - \frac{1}{\alpha^2}\right)\right) = 0$$

$$x^2 - \left(\alpha + \beta - \frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)x + \left(\alpha\beta - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\alpha^2\beta^2}\right) = 0$$

$$x^2 - \left(\alpha + \beta - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\right)x + \left(\alpha\beta + \frac{1}{\alpha^2\beta^2} - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right) = 0$$

$$x^2 - \left(\alpha + \beta - \left(\frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}\right)\right)x + \left(\alpha\beta + \frac{1}{(\alpha\beta)^2} - \left(\frac{\beta + \alpha}{\alpha\beta}\right)\right) = 0$$

$$x^2 - \left(\alpha + \beta - \left(\frac{\beta^2 + \alpha^2}{(\alpha\beta)^2}\right)\right)x + \left(\alpha\beta + \frac{1}{(\alpha\beta)^2} - \left(\frac{\beta + \alpha}{\alpha\beta}\right)\right) = 0$$

$$x^2 - \left(\frac{3}{2} - \left(\frac{\left(-\frac{19}{4}\right)}{\left(\frac{7}{2}\right)^2}\right)\right)x + \left(\frac{7}{2} + \frac{1}{\left(\frac{7}{2}\right)^2} - \left(\frac{\left(\frac{3}{2}\right)}{\left(\frac{7}{2}\right)}\right)\right) = 0$$

$$x^2 - \left(\frac{185}{98}\right)x + \left(\frac{309}{98}\right) = 0$$

$$x^2 - \frac{185}{98}x + \frac{309}{98} = 0$$

$$x^2 - \frac{185}{98}x + \frac{309}{98} = 0 \quad \times 98$$

$$98x^2 - 185x + 309 = 0 \longleftarrow \text{all integer coefficients}$$

$$\boxed{98x^2 - 185x + 309 = 0 \qquad p = 98,\ q = -185,\ r = 309}$$

whenever trig is involved, work in radians

6. (i)

$$f(x) = x - 4 - \cos\left(5\sqrt{x}\right) \qquad x > 0$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval [2.5, 3.5]

(2)

(b) Use linear interpolation once on the interval [2.5, 3.5] to find an approximation to α, giving your answer to 2 decimal places.

(2)

(ii)

$$g(x) = \frac{1}{10}x^2 - \frac{1}{2x^2} + x - 11 \qquad x > 0$$

(a) Determine $g'(x)$.

(2)

The equation $g(x) = 0$ has a root β in the interval [6, 7]

(b) Using $x_0 = 6$ as a first approximation to β, apply the Newton–Raphson procedure once to $g(x)$ to find a second approximation to β, giving your answer to 3 decimal places.

(2)

ia. $f(x) = x - 4 - \cos(5\sqrt{x})$

$f(2.5) = (2.5) - 4 - \cos(5\sqrt{(2.5)}) = -1.448310529$

$f(3.5) = (3.5) - 4 - \cos(5\sqrt{(3.5)}) = 0.4975064212$

There is a sign change from negative to positive, and $f(x)$ is continuous,
$\therefore$ root, α, lies in interval [2.5, 3.5].

b. Linear Interpolation:

$$\alpha = \frac{af(b) - bf(a)}{f(b) - f(a)} \quad \text{for interval } [a,b]$$

where α is the root

$a = 2.5 \qquad b = 3.5$

$$\alpha = \frac{2.5\,f(3.5) - 3.5\,f(2.5)}{f(3.5) - f(2.5)}$$

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Question 6 continued

$f(2.5) = -1.448310529$ } use part (a)

$f(3.5) = 0.4975064212$

$$\alpha = \frac{2.5(0.4975064212) - 3.5(-1.448310529)}{(0.4975064212) - (-1.448310529)}$$

$\alpha = 3.244320029$

$\boxed{\alpha = 3.24 \ (2d.p.)}$

iia. $g(x) = \frac{1}{10}x^2 - \frac{1}{2x^2} + x - 11$

rewrite as indices:

$g(x) = \frac{1}{10}x^2 - \frac{1}{2}x^{-2} + x - 11$

$g'(x) = \frac{1}{5}x + x^{-3} + 1$

$\boxed{g'(x) = \frac{1}{5}x + \frac{1}{x^3} + 1}$

b. $X_0 = 6$

we are looking for X_1

Numerical Sol^{n}s of Eq^{n}s:

Newton-Rhapson iteration for solving $f(x) = 0$

$$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)}$$

From Newton-Rhapson iteration: $X_1 = X_0 - \frac{f(X_0)}{f'(X_0)}$

(1) must find $g(X_0)$ by subbing in X_0 (1.4) into $f(x)$

(2) must find $g'(X_0)$ by differentiating $f(x)$ and subbing in X_0 (1.4) into $f'(x)$.

Question 6 continued

(1) $g(6) = \frac{1}{10}(6)^2 - \frac{1}{2(6)^2} + (6) - 11 = -\frac{509}{360}$

(2) $g'(x) = \frac{1}{5}x + \frac{1}{x^3} + 1$

$g'(6) = \frac{1}{5}(6) + \frac{1}{(6)^3} + 1 = \frac{2321}{1080}$

$$X_1 = (6) - \frac{\left(-\frac{509}{360}\right)}{\left(\frac{2321}{1080}\right)}$$

$X_1 = 6.641327173$

$\boxed{X_1 = 6.641 \ (3d.p.)}$

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7. The parabola C has equation $y^2 = \frac{4}{3}x$

The point $P\left(\frac{1}{3}t^2, \frac{2}{3}t\right)$, where $t \neq 0$, lies on C.

(a) Use calculus to show that the normal to C at P has equation

$$3tx + 3y = t^3 + 2t$$

(3)

The normal to C at the point where $t = 9$ meets C again at the point Q.

(b) Determine the exact coordinates of Q.

(4)

a. Differentiate Parabola w.r.t. x:

$y^2 = \frac{4}{3}x$

$2y\left(\frac{dy}{dx}\right) = \frac{4}{3}$

$\frac{dy}{dx} = \frac{2}{3y}$

Sub in coord. $P\left(\frac{1}{3}t^2, \frac{2}{3}t\right)$

$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{3}t^2, \frac{2}{3}t\right)} \quad \frac{2}{3\left(\frac{2}{3}t\right)} = \frac{2}{2t} = \frac{1}{t}$

$m_{tangent} = \frac{1}{t}$

$m_{normal} = -t$

Set up line eqⁿ $y - y_1 = m(x - x_1)$ with $m = -t$

$(x_1, y_1) = \left(\frac{1}{3}t^2, \frac{2}{3}t\right)$

$y - \frac{2}{3}t = -t\left(x - \frac{1}{3}t^2\right)$

$y - \frac{2}{3}t = -tx + \frac{1}{3}t^3$

×3 on both sides

$3\left(y - \frac{2}{3}t\right) = 3\left(-tx + \frac{1}{3}t^3\right)$

$3y - 2t = -3tx + t^3$

$\boxed{3tx + 3y = t^3 + 2t \text{ // shown}}$

b. Sub in $t = 9$ into normal line eqⁿ: $3(9)x + 3y = (9)^3 + 2(9)$

$27x + 3y = 747$

Question 7 continued

(1): normal line eq^n: $27x + 3y = 747$

(2): hyperbola eq^n: $y^2 = \frac{4}{3}x$

solve these 2 eq^{n}s simultaneously to find points of intersection (when $t=9$ and coord. Q)

(2) $y^2 = \frac{4}{3}x$) $\times \frac{4}{3}$ on both sides

$\frac{3}{4}y^2 = x$) $\times 27$ on both sides

$\frac{81}{4}y^2 = 27x$

(1) $27x + 3y = 747$

$27x = 747 - 3y$

$\frac{81}{4}y^2 = 747 - 3y$) $\times 4$ on both sides

$81y^2 = 2988 - 12y$

$81y^2 + 12y - 2988 = 0$) $\div 3$

$27y^2 + 4y - 996 = 0$

$(27y + 166)(y - 6) = 0$

$y = \frac{-166}{27}$ or $y = 6$

To determine which y-coordinate belongs to coord. Q, find coord at $t=9$ by subbing $t=9$ into $P\left(\frac{1}{3}t^2, \frac{2}{3}t\right)$

$\left(\frac{1}{3}(9)^2, \frac{2}{3}(9)\right) = (27, 6)$

$\therefore y = 6$ belongs to $t=9$

$\therefore y = \frac{-166}{27}$ belongs to coord. Q

find x-coord by subbing back into either (1) or (2) and solving for x.

$y^2 = \frac{4}{3}x$

$\frac{3}{4}y^2 = x$

$\frac{3}{4}\left(\frac{-166}{27}\right)^2 = x = \frac{6889}{243}$

$\boxed{\therefore Q: \left(\frac{6889}{243}, \frac{-166}{27}\right)}$

(Total for Question 7 is 7 marks)

8. (a) Use the standard results for summations to show that, for all positive integers n,

$$\sum_{r=1}^{n} r\left(2r^2 - 3r - 1\right) = \frac{1}{2}n(n+1)^2(n-2)$$

(4)

(b) Hence show that, for all positive integers n,

$$\sum_{r=n}^{2n} r\left(2r^2 - 3r - 1\right) = \frac{1}{2}n(n-1)(an+b)(cn+d)$$

where a, b, c and d are integers to be determined.

(4)

a. $\sum_{r=1}^{n} r(2r^2-3r-1) = \sum_{r=1}^{n} 2r^3 - 3r^2 - r$

can split summation into 3

$\sum_{r=1}^{n} 2r^3 - \sum_{r=1}^{n} 3r^2 - \sum_{r=1}^{n} r$

factor 2 out summation

factor 3 out summation

$2\sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$

Summations

$\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

must learn!

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

in the formulae booklet

$= 2\left[\frac{1}{4}n^2(n+1)^2\right] - 3\left[\frac{1}{6}n(n+1)(2n+1)\right] - \left[\frac{1}{2}n(n+1)\right]$

$= \frac{1}{2}n^2(n+1)^2 - \frac{1}{2}n(n+1)(2n+1) - \frac{1}{2}n(n+1)$

factor out common factor $\frac{1}{2}n(n+1)$

$= \frac{1}{2}n(n+1)\left[n(n+1) - (2n+1) - 1\right]$

$= \frac{1}{2}n(n+1)\left[n^2+n-2n-1-1\right]$

$= \frac{1}{2}n(n+1)\left[n^2-n-2\right]$

$= \frac{1}{2}n(n+1)(n-2)(n+1)$

$\boxed{\frac{1}{2}n(n+1)^2(n-2)}$ // shown

Question 8 continued

b. $\sum_{r=n}^{2n} r(2r^2-3r-1) = \sum_{r=1}^{2n} r(2r^2-3r-1) - \sum_{r=1}^{n-1} r(2r^2-3r-1)$

$= \frac{1}{2}(2n)(2n+1)^2\underbrace{(2n-2)}_{\text{factor 2 out}} - \frac{1}{2}(n-1)(n-1+1)^2(n-1-2)$

$= \cancel{\frac{1}{2}}(2n)(2n+1)^2\cancel{2}(n-1) - \frac{1}{2}(n-1)(n)^2(n-3)$

$= 2n(2n+1)^2(n-1) - \frac{1}{2}(n-1)(n)^2(n-3)$

factor out $\frac{1}{2}n(n-1)$

$= \frac{1}{2}n(n-1)\left[4(2n+1)^2 - n(n-3)\right]$

$= \frac{1}{2}n(n-1)\left[4(4n^2+4n+1) - n^2+3n\right]$

$= \frac{1}{2}n(n-1)\left[16n^2+16n+4-n^2+3n\right]$

$= \frac{1}{2}n(n-1)\left[15n^2+19n+4\right]$

$= \frac{1}{2}n(n-1)(15n+4)(n+1)$

$\boxed{\frac{1}{2}n(n-1)(15n+4)(n+1) \quad a=15, b=4, c=1, d=1}$

DO NOT WRITE IN THIS AREA

9. Given that

$$\frac{3z-1}{2} = \frac{\lambda+5\mathrm{i}}{\lambda-4\mathrm{i}}$$

where λ is a real constant,

(a) determine z, giving your answer in the form $x + y\mathrm{i}$, where x and y are real and in terms of λ.

(4)

Given also that $\arg z = \frac{\pi}{4}$

(b) find the possible values of λ.

(2)

a. rewrite $\frac{\lambda+5i}{\lambda-4i}$ in form $a+bi$:

multiply top and bottom by complex conjugate of denominator

$$\frac{\lambda+5i}{\lambda-4i} \times \frac{\lambda+4i}{\lambda+4i} = \frac{\lambda^2+4i\lambda+5i\lambda+20i^2}{\lambda^2-4i\lambda+4i\lambda-16i^2}$$

$$= \frac{\lambda^2+9i\lambda+20i^2}{\lambda^2-16i^2}$$

$$= \frac{\lambda^2+9i\lambda+20(-1)}{\lambda^2-16(-1)}$$

$$= \frac{\lambda^2+9i\lambda-20}{\lambda^2+16}$$

$$\frac{3z-1}{2} = \frac{\lambda^2+9i\lambda-20}{\lambda^2+16}$$

rearrange for z

$$z = \frac{2\left(\frac{\lambda^2+9i\lambda-20}{\lambda^2+16}\right)+1}{3}$$

$$z = \frac{\frac{2(\lambda^2+9i\lambda-20)}{\lambda^2+16}+1}{3} = \frac{\frac{2\lambda^2+18i\lambda-40}{\lambda^2+16}+\frac{\lambda^2+16}{\lambda^2+16}}{3} = \frac{\frac{2\lambda^2+18i\lambda-40+\lambda^2+16}{\lambda^2+16}}{3}$$

Question 9 continued

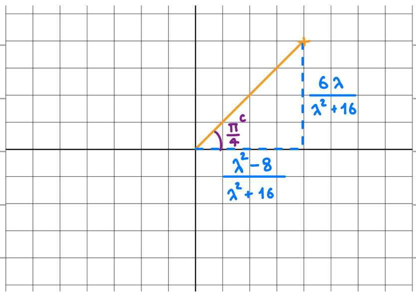
$$= \frac{\dfrac{2\lambda^2 + 18i\lambda - 40 + \lambda^2 + 16}{\lambda^2 + 16}}{3} = \frac{3\lambda^2 - 24 + 18i\lambda}{\lambda^2+16} \div 3 = \frac{\cancel{3}(\lambda^2 - 8\lambda + 6i\lambda)}{\lambda^2+16} \times \frac{1}{\cancel{3}}$$

$$= \frac{\lambda^2 - 8 + 6i\lambda}{\lambda^2+16}$$

$$= \frac{\lambda^2 - 8}{\lambda^2+16} + \frac{6\lambda}{\lambda^2+16}i$$

$$\boxed{z = \frac{\lambda^2 - 8}{\lambda^2+16} + \frac{6\lambda}{\lambda^2+16}i}$$

b.



$$\tan\left(\frac{\pi}{4}\right) = \frac{\dfrac{6\lambda}{\cancel{\lambda^2+16}}}{\dfrac{\lambda^2-8}{\cancel{\lambda^2+16}}}$$

$$1 = \frac{6\lambda}{\lambda^2 - 8}$$

$$\lambda^2 - 8 = 6\lambda$$

$$\lambda^2 - 6\lambda - 8 = 0$$

$$(\lambda - 3)^2 - 9 - 8 = 0$$

$$(\lambda - 3)^2 - 17 = 0$$

$$(\lambda - 3)^2 = 17$$

$$\lambda - 3 = \pm\sqrt{17}$$

$$\lambda = 3 \pm \sqrt{17}$$

As both real and imaginary component must be positive, $\lambda = 3 - \sqrt{17}$ cannot be a solⁿ. $\therefore \lambda = 3 + \sqrt{17}$ only.

$$\boxed{\lambda = 3 + \sqrt{17}}$$

(Total for Question 9 is 6 marks)

10. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^n = 3^{n-1}\begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}$$

(5)

(ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 8^{2n+1} + 6^{2n-1}$$

is divisible by 7

(5)

i. 1) Base Case : Prove true for n = 1

LHS : $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^1 = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$

RHS : $3^{1-1}\begin{bmatrix} 2(1)+3 & -(1) \\ 4(1) & 3-2(1) \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$

LHS = RHS $\therefore$ true for n = 1

(2) Assume true for n = k :

$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^k = 3^{k-1}\begin{bmatrix} 2k+3 & -k \\ 4k & 3-2k \end{bmatrix}$

(3) Prove true for n = k + 1 :

$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^k \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$

Thinking ahead, sub in n = k + 1 into proof : this is what we are trying to prove using (2) :

$\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^{k+1} = 3^{k+1-1}\begin{pmatrix} 2(k+1)+3 & -(k+1) \\ 4(k+1) & 3-2(k+1) \end{pmatrix}$

Make a note so we know what we are working towards

$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^{k+1} = 3^{k-1}\begin{bmatrix} 2k+3 & -k \\ 4k & 3-2k \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$

$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^{k+1} = 3^{k-1}\begin{bmatrix} (2k+3)(5)+(4)(-k) & (2k+3)(-1)+(-k)(1) \\ (4k)(5)+(3-2k)(4) & (4k)(-1)+(3-2k)(1) \end{bmatrix}$

$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^{k+1} = 3^{k-1}\begin{bmatrix} 6k+15 & -3k-3 \\ 12k+12 & -6k+3 \end{bmatrix}$

Question 10 continued

$$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^{k+1} = 3^{k-1}\begin{bmatrix} 3(2k+5) & -3(k+1) \\ 3(4k+4) & 3(-2k+1) \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^{k+1} = 3^{k-1} \times 3^{1}\begin{bmatrix} (2k+5) & -(k+1) \\ (4k+4) & (-2k+1) \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}^{k+1} = 3^{k+1-1}\begin{bmatrix} 2(k+1)+3 & -(k+1) \\ 4(k+1) & 3-2(k+1) \end{bmatrix}$$ $\therefore$ true for $n=k+1$

(4) Conclusion

If this result is true for $n=k$, then it is true for $n=k+1$. As the result has been shown to be true for $n=1$, then the result is true for all n.

ii. 1) Base Case : Prove true for $n=1$

$f(1) = 8^{2(1)+1} + 6^{2(1)-1}$

$= 8^3 + 6^1$

$= 518$

$= 74(7)$ $\leftarrow$ divisible by 7

$\therefore$ true for $n=1$

(2) Assume true for $n=k$:

$f(k) = 8^{2k+1} + 6^{2k-1}$

(3) Prove true for $n=k+1$:

$f(k+1) = 8^{2(k+1)+1} + 6^{2(k+1)-1}$

$= 8^{2k+2+1} + 6^{2k+2-1}$

$= 8^{2k+3} + 6^{2k+1}$

Question 10 continued

$f(k+1) - f(k) = \left[8^{2k+3} + 6^{2k+1}\right] - \left[8^{2k+1} + 6^{2k-1}\right]$

$= \left[8^2(8^{2k+1}) + 6^2(6^{2k-1})\right] - \left[8^{2k+1} + 6^{2k-1}\right]$

$= \left[64(8^{2k+1}) + 36(6^{2k-1})\right] - \left[8^{2k+1} + 6^{2k-1}\right]$

$= 64(8^{2k+1}) + 36(6^{2k-1}) - (8^{2k+1}) - (6^{2k-1})$

$= 63(8^{2k+1}) + 35(6^{2k-1})$

$= 28(8^{2k+1}) + 35(8^{2k+1}) + 35(6^{2k-1})$

$= 28(8^{2k+1}) + 35\left[8^{2k+1} + 6^{2k-1}\right]$

$= 28(8^{2k+1}) + 35f(k)$

$f(k+1) - f(k) = 28(8^{2k+1}) + 35f(k)$

28 is divisible by 7

$f(k+1) = 28(8^{2k+1}) + 36f(k)$

$f(k+1) = 7 \times 4(8^{2k+1}) + 36f(k)$ $\therefore$ true for $n = k+1$

(4) Conclusion

If this result is true for $n = k$, then it is true for $n = k+1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.